Table 1 illustrates an example of sampled data built by five points, which could have been obtained experimentally, for example, as tables 4 and 5 in part II of the project.

<table>
<thead>
<tr>
<th>(x_i)</th>
<th>(f(x_i))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.51</td>
</tr>
<tr>
<td>1</td>
<td>3.60</td>
</tr>
<tr>
<td>2</td>
<td>10.40</td>
</tr>
<tr>
<td>3</td>
<td>21.60</td>
</tr>
<tr>
<td>4</td>
<td>36.50</td>
</tr>
</tbody>
</table>

The aim of this example is to find a polynomial function of second order (parabolic function) of the type

\[
f(x) = c_2x^2 + c_1x + c_0
\]  

which fits the 5 sampled points. The polynomial function should be chosen in such a way that the sum of the square of the errors \(\epsilon_i^2\) \((i = 1, ..., 5)\) among the parabolic function \(f(x)\) and the 5 points is the smallest value. In other words, the goal of this example is to identify the values of \(c_0, c_1\) and \(c_2\) by minimizing the sum of the square of the errors \(\sum_5 \epsilon_i^2 = \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 + \epsilon_5^2\) among \(f(x)\) and the 5 points presented in table 1.

**Solution:** You have to find 3 unknowns, i.e. \(c_0, c_1\) and \(c_2\), but you have 5 equations. Actually you have more equations than unknowns, and it is difficult to satisfy all them simultaneously. Thus, when you find a set of \(c_0, c_1\) and \(c_2\) you will have an error, which shall be as small as possible. Considering the errors \(\epsilon_i\) \((i = 1, ..., 5)\) among the polynomial function and the five points, you can write the 5 equations as follow:

\[
\begin{align*}
c_2x_1^2 + c_1x_1 + c_0 &= f(x_1) + \epsilon_1 \\
c_2x_2^2 + c_1x_2 + c_0 &= f(x_2) + \epsilon_2 \\
c_2x_3^2 + c_1x_3 + c_0 &= f(x_3) + \epsilon_3 \\
c_2x_4^2 + c_1x_4 + c_0 &= f(x_4) + \epsilon_4 \\
c_2x_5^2 + c_1x_5 + c_0 &= f(x_5) + \epsilon_5
\end{align*}
\]  

(2)

Writing the 5 equations in matrix form, you get:

\[
\begin{bmatrix}
  x_1^2 & x_1 & 1 \\
  x_2^2 & x_2 & 1 \\
  x_3^2 & x_3 & 1 \\
  x_4^2 & x_4 & 1 \\
  x_5^2 & x_5 & 1
\end{bmatrix}
\begin{bmatrix}
  c_2 \\
  c_1 \\
  c_0
\end{bmatrix}
= 
\begin{bmatrix}
  f(x_1) \\
  f(x_2) \\
  f(x_3) \\
  f(x_4) \\
  f(x_5)
\end{bmatrix}
+ 
\begin{bmatrix}
  \epsilon_1 \\
  \epsilon_2 \\
  \epsilon_3 \\
  \epsilon_4 \\
  \epsilon_5
\end{bmatrix}
\]  

(3)
The error represented by the vector \( \epsilon \) is a function of vector \( c \) and can be written as

\[ \epsilon = Ac - b \]  

(5)

When the error is written in vector form, the sum of the square of the errors can be written as

\[ \epsilon^T \epsilon = \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 + \epsilon_5^2 = \sum_{i=1}^{5} \epsilon_i^2 \]  

(6)

or, based on equation (5), using the vector \( c \) as

\[ \epsilon^T \epsilon = (Ac - b)^T(Ac - b) \]  

(7)

"Minimizing the sum of the square of the errors in relation to vector \( c \)" means

\[ \frac{\partial}{\partial c} (\epsilon^T \epsilon) = 0 \Rightarrow \frac{\partial}{\partial c} [(Ac - b)^T(Ac - b)] = 0 \Rightarrow A^T(Ac - b) + (Ac - b)^T A = 0 \]  

(8)

what leads to

\[ c = (A^T A)^{-1} A^T b \]  

(9)

Thus, the vector \( c \), calculated based on equation (9), minimizes the sum of the square of the errors among the 5 points and the function \( f(x) \). Following, a summary of the calculation is presented, and in figure 1 the mathematical meaning of the Least Square Method is illustrated.
\[
A^T A = \begin{bmatrix}
354 & 100 & 30 \\
100 & 30 & 10 \\
30 & 10 & 5
\end{bmatrix}
\] (13)

\[
(A^T A)^{-1} = \begin{bmatrix}
0.0714 & -0.2857 & 0.1429 \\
-0.2857 & 1.2429 & -0.7714 \\
0.1429 & -0.7714 & 0.8857
\end{bmatrix}
\] (14)

\[
(A^T A)^{-1} A^T = \begin{bmatrix}
0.1429 & -0.0714 & -0.1429 & -0.0714 & 0.1429 \\
-0.7714 & 0.1857 & 0.5714 & 0.3857 & -0.3714 \\
0.8857 & 0.2571 & -0.0857 & -0.1429 & 0.0857
\end{bmatrix}
\] (15)

\[
c = (A^T A)^{-1} A^T b = \begin{bmatrix}
2.0014 \\
0.9923 \\
0.5289
\end{bmatrix}
\] (16)

• **SOLUTION:** Substituting the values \(c_2 = 2.0014, c_1 = 0.9923\) and \(c_0 = 0.5289\) into equation (1) the polynomial function becomes:

\[
f(x) = 2.0014 \cdot x^2 + 0.9923 \cdot x + 0.5289
\] (17)

• **ERRORS:** The errors \(\epsilon^T = \{ \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5 \}\) among the five points presented in table 1 and the polynomial function \(f(x) = 2.0014 \cdot x^2 + 0.9923 \cdot x + 0.5289\) are:

\[
\epsilon = Ac - b = \begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3 \\
\epsilon_4 \\
\epsilon_5
\end{bmatrix} = \begin{bmatrix}
0.0189 \\
-0.0774 \\
0.1191 \\
-0.0816 \\
0.0205
\end{bmatrix}
\] (18)

• **MATHEMATICAL MEANING:** The mathematical meaning of the Least-Squares Method is illustrated in figure 1. By using the function \(f(x)\) the value of the five points can be predicted with small errors. For example, point 3 in table 1 \(x_3 = 2\) can be predicted by the function \(f(2) = 2.0014 \cdot 2^2 + 0.9923 \cdot 2 + 0.5289 = 10.5191\). The error between the function and point 3 is \(\epsilon_3 = f(2) - 10.40 = 10.5191 - 10.40 = 0.1191\). Such an error \(\epsilon_3\) and the others are calculated in equation 18. Error \(\epsilon_3\) is also illustrated in the zoom of figure 1.
Function $f(x)$ obtained by using Least-Square Method and the five points * given in table 1 – the sum of the square of the errors $\epsilon_i^2$ ($i = 1, \ldots, 5$) among the parabolic function $f(x)$ and the 5 points is the minimum value.

$$f(x) = 2.0014x^2 + 0.9923x + 0.5289$$