41514 – Dynamics of Machinery
– Theory, Experiment, Phenomenology and Industrial Applications –

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1. Recapitulation – Mathematical Modeling & Steps
2. Example – System of Particle
3. Example – Rigid Body (Top)
4. Example – Linearization & Application to Rotating Machines
1. Recapitulation – Mathematical Modeling & Steps

**Physical System**

- Assumptions (simplifications)

**Mechanical Model**

- Newton, Euler, D'Alembert, Lagrange, Hamilton, Jourdain (principles & axioms)

**Mathematical Model**

\[ \ddot{x}(t) = f(\dot{x}(t), x(t)) \]

**Static Equilibrium Position** (Linearization)

\[ M \ddot{x}(t) + D \dot{x}(t) + K x(t) = f(t) \] (structure)

\[ M \ddot{x}(t) + (D + G(\Omega)) \dot{x}(t) + K x(t) = f(t) \] (machine)

**Solution:**

\[ x(t) = \sum C_i u_i e^{\lambda_i t} + x_p(t) \]

\[ \lambda(\Omega) = [\lambda_1, \lambda_2, \lambda_3, \ldots] \] (eigenvalues)

\[ U(\Omega) = [u_1, u_2, u_3, \ldots] \] (eigenvectors)

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*Mass Elements M*

- Particle
- Rigid Body
- Distributed

*Spring Elements K*

- Elasticity Theory & Material
- Magnetism
- Fluid Mechanics

*Damping Elements D*

- Fluid Mechanics
- Contact Mechanics (friction)
1. Recapitulation – Mathematical Modeling & Steps

Physical System

Assumptions (simplifications)

Mechanical Model

Newton, Euler, D’Alembert, Lagrange, Hamilton, Jourdain (principles & axioms)

Mathematical Model

• Kinematics
  1. Reference Frames (systems of coordinates)
  2. Transformation Matrices
  3. Position Vectors
  4. Velocity Vectors (linear and angular)
  5. Acceleration Vectors (linear and angular)

• Dynamics
  6. Mass Properties (mass center, moments of inertia)
  7. Force and Moment Vectors
  8. Dynamic Equilibrium:
     (Newton, Euler, Lagrange …)
  9. Equilibrium Poition (Linearization, Vibration Analysis)

\[ \ddot{x}(t) = f(\dot{x}(t), x(t)) \]
\[ M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = f(t) \]

* Mass Elements M
  Particle
  Rigid Body
  Distributed

* Spring Elements K
  Elasticity Theory & Material
  Magnetism
  Fluid Mechanics

* Damping Elements D
  Fluid Mechanics
  Contact Mechanics (friction)
1. Recapitulation (Kinematics)

\[ \dot{\mathbf{r}}_B = \dot{\mathbf{r}}_A + \dot{\mathbf{\omega}} \times \mathbf{r}_{AB} + \dot{\mathbf{v}}_{\text{rel}} \] (1.8)

or

\[ \dot{\mathbf{r}}_B = \dot{\mathbf{r}}_A + \dot{\mathbf{\omega}} \times \left( \mathbf{T}^T_{\theta} \cdot \mathbf{r}_{AB} \right) + \mathbf{T}^T_{\theta} \cdot \frac{d}{dt} \left( \mathbf{r}_{AB} \right) + \dot{\mathbf{v}}_{\text{rel}} \]

- \( \dot{\mathbf{r}}_A \) is the absolute linear velocity of point A (point where the moving reference system is located) represented in the inertial reference system I.

- \( \dot{\mathbf{\omega}} \times \mathbf{r}_{AB} \) is the cross product between the angular velocity vector of the moving reference system and the position vector \( \mathbf{r}_{AB} \), both of them described in the inertial reference frame. The vector \( \mathbf{r}_{AB} \) has its origin in point A and points toward the studied point B. It must be stressed that \( \mathbf{r}_{AB} \) is usually described in the moving reference system \( B_1 \), in order to simplify the description of the movement. When this vector is multiplied by the coordinate transformation matrix \( \mathbf{T}^T_{\theta} \), one obtains its representation in the inertial system.

- \( \dot{\mathbf{v}}_{\text{rel}} \) is the relative velocity of point B with respect to point A. This velocity is obtained by deriving the position vector \( \mathbf{r}_{AB} \) with respect to time, when it is represented in the moving reference system, which absolute angular velocity is \( \dot{\mathbf{\omega}} \). Deriving this vector in the moving system, entails obtaining its representation in the moving system \( B_1 \). Moreover, deriving this vector in the moving system, entails that you obtain a relative quantity, i.e. \( \dot{\mathbf{v}}_{\text{rel}} \). Then, it is necessary to multiply this vector by the transformation matrix \( \mathbf{T}^T_{\theta} \) to obtain its representation in the inertial reference frame. Afterwards, it is possible to obtain the sum of all the terms in the eq.(1.8) expressed in the same reference frame.

The reader must focus on an important fact: many students are confused between the concept of representing the absolute velocity in the moving system and the concept of relative velocity vector. It must be stressed that the absolute velocity vector will always be an absolute velocity vector. Any vector can be expressed both in an inertial system as in a moving reference system, hence the absolute velocity vector represented in the moving reference frame will still be the absolute velocity vector. A relative velocity vector can also be represented in a moving or in an inertial reference frame, but expressing it into the inertial system does not imply that it is transformed into an absolute velocity vector.

Frequently, it is more convenient to represent the velocities in the moving reference frame \( B_1 \) and equation (1.8) can be rewritten as:

\[ \mathbf{v}_{B_1} = \mathbf{v}_A + \mathbf{\omega} \times \mathbf{r}_{AB} + \frac{d}{dt} \left( \mathbf{r}_{AB} \right) \] (1.9)

- **Position Vector**

\[ \mathbf{r}_{OB} = \mathbf{r}_{OA} + \mathbf{T}^T_{\theta} \cdot \mathbf{r}_{AB} = \mathbf{r}_{OA} + \mathbf{r}_{AB} \]

- **Velocity Vector**
1. Recapitulation (Kinematics)

- **Acceleration Vector**

\[
B_1a_B = B_1a_A + B_1\dot{\omega} \times B_1r_{AB} + B_1\omega \times (B_1\omega \times B_1r_{AB}) + 2B_1\omega \times \frac{d}{dt}(B_1r_{AB}) + \frac{d^2}{dt^2}(B_1r_{AB})
\]

or

\[
\dot{a}_B = \dot{a}_A + \dot{\omega} \times r_{AB} + \omega \times (\omega \times r_{AB}) + 2\omega \times \left( T^T_{A0} \cdot \frac{d}{dt}(n_1r_{AN}) \right) + T^T_{A0} \cdot \frac{d^2}{dt^2}(n_1r_{AN})
\]

- \( \dot{a}_A \) is the **absolute** linear acceleration vector of the point \( A \), where the origin of the moving reference system is located, represented in the inertial reference system \( I \).

- \( \dot{\omega} \times r_{AB} \) the cross product between the **absolute** angular acceleration of the moving reference frame and the position vector \( r_{AB} \), both of them described in the inertial system. The vector \( \dot{r}_{AB} \) has its origin in the point \( A \) and it is pointing toward the studied point \( B \). This term is directly related to the tangential acceleration, due to the absolute variation on time of the vector \( \dot{\omega} \).

- \( \omega \times (\omega \times r_{AN}) \) is the cross product between the **absolute** angular velocity vector of the moving reference system and the vector resulting from the operation \( \omega \times r_{AN} \). This vector is related to the change in direction of the vector \( \omega \times r_{AB} \). The vector \( \omega \times r_{AN} \) rotates with the angular speed \( \dot{\omega} \).

- \( 2\omega \times \dot{v}_{BA} \) is the cross product of the absolute angular velocity vector of the moving reference system and the relative velocity of point \( B \) with respect to point \( A \), both of them expressed in the inertial system \( I \). This term is known as the Coriolis acceleration, expressing the change in direction of the relative velocity vector \( \dot{v}_{BA} \). This vector rotates in space with an angular velocity \( \dot{\omega} \). Example 5 of this chapter deals with the mathematical and physical meaning of this term.

- \( a_{BA} \) is the relative acceleration of point \( B \) with respect to point \( A \) (origin of the moving reference system). This acceleration is obtained when deriving twice the position vector \( n_1r_{AN} \) with respect to time, represented in the moving reference system, whose angular velocity is given by \( \dot{\omega} \). By deriving this vector in the moving system, one obtains its representation in the moving system \( B1 \). Then, it is necessary to multiply it by the transformation matrix \( T^T_A \) to obtain the representation of this vector in the inertial system, in order to obtain the sum of all the terms of eq.(1.10) expressed in the same base.
2. Example – System of Particles (2 particles in 2D)

Free-Body Diagram

Two Angular Motions – two rotations around Z
2. Example – System of Particles (2 particles in 2D) (Kinematics)

Acceleration Vectors

- **Particle B**

\[ B_1^a_B = \frac{B_1 a_0}{=0} + B_1 \omega_1 \times B_1 \omega_1 \times B_1 l_1 + B_1 \dot{\omega}_1 \times B_1 l_1 + 2 \cdot B_1 \omega_1 \times B_1 v_{rel} + B_1 a_{rel} \]

\[ B_1^a_B = B_1 \omega_1 \times B_1 \omega_1 \times B_1 l_1 + B_1 \dot{\omega}_1 \times B_1 l_1 = \begin{bmatrix} -l_1 \ddot{\theta}_1 \\ -l_1 \dot{\theta}_1^2 \\ 0 \end{bmatrix} \]

- **Particle A**

\[ B_2^a_A = B_2 a_B + B_2 \omega_2 \times B_2 \omega_2 \times B_2 l_2 + B_2 \dot{\omega}_2 \times B_2 l_2 + 2 \cdot B_2 \omega_2 \times B_2 v_{rel} + B_2 a_{rel} \]

\[ B_2^a_A = T_{\theta_2} B_1 a_B + B_2 \omega_2 \times B_2 \omega_2 \times B_2 l_2 + B_2 \dot{\omega}_2 \times B_2 l_2 \]

\[ B_2^a_A = \begin{bmatrix} -l_1(\ddot{\theta}_1 \cos \theta_2 + \dot{\theta}_1^2 \sin \theta_2) - l_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ l_1(\ddot{\theta}_1 \sin \theta_2 - \dot{\theta}_1^2 \cos \theta_2) - l_2(\ddot{\theta}_1 + \ddot{\theta}_2)^2 \\ 0 \end{bmatrix} \]
2. Example – System of Particles (2 particles in 2D)

- **Force Vectors: Reaction and External Applied Forces**

\[ \begin{align*}
\text{Particle B:} & \quad \sum_{B_1} \mathbf{F}_B = m_{B_1} a_{B_1} \quad \Rightarrow \quad B_1 \mathbf{P}_B + B_1 \mathbf{T}_1 - B_1 \mathbf{T}_2 = m_{B_1} a_{B_1} \\
& \quad \begin{cases} m_1 g \sin \theta_1 - T_2 \sin \theta_2 \\ m_1 g \cos \theta_1 - T_1 + T_2 \cos \theta_2 \\ 0 \end{cases} = m_1 \begin{cases} -l_1 \ddot{\theta}_1 \\ -l_1 \dot{\theta}_1^2 \\ 0 \end{cases} \\
\text{Particle A:} & \quad \sum_{B_2} \mathbf{F}_A = m_{B_2} a_{B_2} \quad \Rightarrow \quad B_2 \mathbf{P}_A + B_2 \mathbf{T}_2 = m_{B_2} a_{B_2} \\
& \quad \begin{cases} m_2 g \sin(\theta_1 + \theta_2) \\ m_2 g \cos(\theta_1 + \theta_2) - T_2 \end{cases} = m_2 \begin{cases} -l_1 (\ddot{\theta}_1 \cos \theta_2 + \dot{\theta}_1^2 \sin \theta_2) - l_2 (\ddot{\theta}_1 + \dot{\theta}_2) \\ l_1 (\ddot{\theta}_1 \sin \theta_2 - \dot{\theta}_1^2 \cos \theta_2) - l_2 (\ddot{\theta}_1 + \dot{\theta}_2)^2 \\ 0 \end{cases}
\end{align*} \]
2. Example – System of Particles (2 particles in 2D)

**Dynamic Reaction Forces** and Equations of Motion

\[
\begin{bmatrix}
0 & -\sin \theta_2 & m_1 l_1 & 0 \\
-1 & \cos \theta_2 & 0 & 0 \\
0 & 0 & m_2 (l_1 \cos \theta_2 + l_2) & m_2 l_2 \\
0 & -1 & -m_2 l_1 \sin \theta_2 & 0 \\
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2 \\
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
-m_1 g \sin \theta_1 \\
-m_1 l_1 \dot{\theta}_1^2 - m_1 g \cos \theta_1 \\
-m_2 [g \sin (\theta_1 + \theta_2) + l_1 \dot{\theta}_1^2 \sin \theta_2] \\
-m_2 [g \cos (\theta_1 + \theta_2) + l_1 \dot{\theta}_1^2 \cos \theta_2 + l_2 (\dot{\theta}_1 + \dot{\theta}_2)^2] \\
\end{bmatrix}
\]

**Cramer’s Rule**

\[
T_1 = \frac{
\begin{vmatrix}
-m_1 g \sin \theta_1 & -\sin \theta_2 & m_1 l_1 & 0 \\
-m_1 l_1 \dot{\theta}_1^2 - m_1 g \cos \theta_1 & \cos \theta_2 & 0 & 0 \\
-m_2 [g \sin (\theta_1 + \theta_2) + l_1 \dot{\theta}_1^2 \sin \theta_2] & 0 & m_2 (l_1 \cos \theta_2 + l_2) & m_2 l_2 \\
-m_2 [g \cos (\theta_1 + \theta_2) + l_1 \dot{\theta}_1^2 \cos \theta_2 + l_2 (\dot{\theta}_1 + \dot{\theta}_2)^2] & -1 & -m_2 l_1 \sin \theta_2 & 0 \\
\end{vmatrix}
}{
\begin{vmatrix}
0 & -\sin \theta_2 & m_1 l_1 & 0 \\
-1 & \cos \theta_2 & 0 & 0 \\
0 & 0 & m_2 (l_1 \cos \theta_2 + l_2) & m_2 l_2 \\
0 & -1 & -m_2 l_1 \sin \theta_2 & 0 \\
\end{vmatrix}
}
\]

\[
T_1 = (\dot{\theta}_1\dot{\theta}_1 l_1^2 m_1^2 m_2 - g l_1 l_2 m_1^2 m_2 \cos \theta_1 - \dot{\theta}_1^2 l_1^2 m_1 m_2^2 \cos \theta_1 - 2 \dot{\theta}_1 \dot{\theta}_2 l_1 l_2 m_1 m_2^2 \cos \theta_2 - \\
\dot{\theta}_2^2 l_1 l_2 m_1 m_2^2 \cos \theta_2 - \dot{\theta}_1^2 l_2^2 m_1 m_2^2 \cos \theta_2 - g l_1 l_2 m_1 m_2^2 \cos \theta_1 \cos (\theta_1 + \theta_2) - \\
g l_1 l_2 m_1 m_2^2 \cos \theta_2 \sin \theta_1 \sin \theta_2 - \dot{\theta}_1^2 l_2^2 m_1 m_2^2 \sin^2 \theta_2 - g l_1 l_2 m_1 m_2^2 \cos \theta_1 \sin^2 \theta_2) / \\
(-l_1 l_2 m_1 m_2 - l_1 l_2 m_2^2 \sin^2 \theta_2)
\]
2. Example – System of Particles (2 particles in 2D)

(Dynamics)

- Dynamic Reaction Forces and Equations of Motion

\[
\begin{bmatrix}
0 & -\sin \theta_2 & m_1 l_1 & 0 \\
-1 & \cos \theta_2 & 0 & 0 \\
0 & 0 & m_2(l_1 \cos \theta_2 + l_2) & m_2 l_2 \\
0 & -1 & -m_2 l_1 \sin \theta_2 & 0
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2 \\
\dot{\theta}_1 \\
\dot{\theta}_2
\end{bmatrix}
= \begin{bmatrix}
-m_1 g \sin \theta_1 \\
-m_1 l_1 \dot{\theta}_1^2 - m_1 g \cos \theta_1 \\
-m_2[g \sin(\theta_1 + \theta_2) + l_1 \dot{\theta}_1^2 \sin \theta_2] \\
-m_2[g \cos(\theta_1 + \theta_2) + l_1 \dot{\theta}_1^2 \cos \theta_2 + l_2(\dot{\theta}_1 + \dot{\theta}_2)^2]
\end{bmatrix}
\]

- Cramer’s Rule

\[
T_2 = \frac{
\begin{vmatrix}
0 & -m_1 g \sin \theta_1 & m_1 l_1 & 0 \\
-1 & -m_1 l_1 \dot{\theta}_1^2 - m_1 g \cos \theta_1 & 0 & 0 \\
0 & -m_2[l_1 \sin(\theta_1 + \theta_2) + l_1 \dot{\theta}_1 \sin \theta_2] & m_2(l_1 \cos \theta_2 + l_2) & m_2 l_2 \\
0 & -m_2[l_1 \cos(\theta_1 + \theta_2) + l_1 \dot{\theta}_1 \cos \theta_2 + l_2(\dot{\theta}_1 + \dot{\theta}_2)^2] & -m_2 l_1 \sin \theta_2 & 0
\end{vmatrix}
}{\begin{vmatrix}
0 & -\sin \theta_2 & m_1 l_1 & 0 \\
-1 & \cos \theta_2 & 0 & 0 \\
0 & 0 & m_2(l_1 \cos \theta_2 + l_2) & m_2 l_2 \\
0 & -1 & -m_2 l_1 \sin \theta_2 & 0
\end{vmatrix}}
\]

\[
T_2 = (-\dot{\theta}_1^2 l_1 l_2 m_1 m_2^2 - 2\dot{\theta}_1 \dot{\theta}_2 l_1 l_2 m_1 m_2 - \dot{\theta}_2^2 l_1 l_2 m_1 m_2 - \ddot{\theta}_1^2 l_1 l_2 m_1 m_2^2 \cos \theta_2 -
2l_1 m_1 m_2^2 \cos(\theta_1 + \theta_2) - 2l_1 m_1 m_2^2 \sin \theta_1 \sin \theta_2) / (-l_1 l_2 m_1 m_2 - l_1 l_2 m_2^2 \sin^2 \theta_2)
\]
2. Example – System of Particles (2 particles in 2D)  

(Dynamics)

- Dynamic Reaction Forces and Equations of Motion

\[
\begin{bmatrix}
0 & -\sin \theta_2 & m_1 l_1 & 0 \\
-1 & \cos \theta_2 & 0 & 0 \\
0 & 0 & m_2(l_1 \cos \theta_2 + l_2) & m_2 l_2 \\
0 & -1 & -m_2 l_1 \sin \theta_2 & 0 \\
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2 \\
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
-m_1 g \sin \theta_1 \\
-m_1 l_1 \dot{\theta}_1^2 - m_1 g \cos \theta_1 \\
-m_2 [g \sin(\theta_1 + \theta_2) + l_1 \dot{\theta}_1^2 \sin \theta_2] \\
-m_2 [g \cos(\theta_1 + \theta_2) + l_1 \dot{\theta}_1^2 \cos \theta_2 + l_2(\dot{\theta}_1 + \dot{\theta}_2)^2] \\
\end{bmatrix}
\]

- Cramer’s Rule

\[
\ddot{\theta}_1 = \frac{\begin{vmatrix}
0 & -\sin \theta_2 & -m_1 g \sin \theta_1 & 0 \\
-1 & \cos \theta_2 & -m_1 l_1 \dot{\theta}_1^2 - m_1 g \cos \theta_1 & 0 \\
0 & 0 & -m_2 [g \sin(\theta_1 + \theta_2) + l_1 \dot{\theta}_1^2 \sin \theta_2] & m_2 l_2 \\
0 & -1 & -m_2 [g \cos(\theta_1 + \theta_2) + l_1 \dot{\theta}_1^2 \cos \theta_2 + l_2(\dot{\theta}_1 + \dot{\theta}_2)^2] & 0 \\
\end{vmatrix}}{\begin{vmatrix}
0 & -\sin \theta_2 & m_1 l_1 & 0 \\
-1 & \cos \theta_2 & 0 & 0 \\
0 & 0 & m_2(l_1 \cos \theta_2 + l_2) & m_2 l_2 \\
0 & -1 & -m_2 l_1 \sin \theta_2 & 0 \\
\end{vmatrix}}
\]

\[
\ddot{\theta}_1 = (gl_2 m_1 m_2 \sin \theta_1 - \dot{\theta}_1^2 l_2 m_2^2 \sin \theta_2 - 2\dot{\theta}_1 \dot{\theta}_2 l_2^2 m_2^2 \sin \theta_2 - \dot{\theta}_2^2 l_2^2 m_2^2 \sin \theta_2 - \\
\dot{\theta}_1^2 l_1 l_2 m_2^2 \cos \theta_2 \sin \theta_2 - gl_2 m_2^2 \cos(\theta_1 + \theta_2) \sin \theta_2)/(l_1 l_2 m_1 m_2 - l_1 l_2 m_2^2 \sin^2 \theta_2)
\]
2. Example – System of Particles (2 particles in 2D)

(Dynamics)

- **Dynamic Reaction Forces and Equations of Motion**

\[
\begin{bmatrix}
0 & -\sin \theta_2 & m_1 l_1 & 0 \\
-1 & \cos \theta_2 & 0 & 0 \\
0 & 0 & m_2 (l_1 \cos \theta_2 + l_2) & m_2 l_2 \\
0 & -1 & -m_2 l_1 \sin \theta_2 & 0 \\
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2 \\
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\end{bmatrix} =
\begin{bmatrix}
-m_1 g \sin \theta_1 \\
-m_1 l_1 \dot{\theta}_1^2 - m_1 g \cos \theta_1 \\
-m_2 [g \sin(\theta_1 + \theta_2) + l_1 \dot{\theta}_1^2 \sin \theta_2] \\
-m_2 [g \cos(\theta_1 + \theta_2) + l_1 \dot{\theta}_1^2 \cos \theta_2 + l_2 (\dot{\theta}_1 + \dot{\theta}_2)^2]
\end{bmatrix}
\]

- **Cramer’s Rule**

\[
\ddot{\theta}_2 = \begin{vmatrix}
0 & -\sin \theta_2 & m_1 l_1 & -m_1 g \sin \theta_1 \\
-1 & \cos \theta_2 & 0 & -m_1 l_1 \dot{\theta}_1^2 - m_1 g \cos \theta_1 \\
0 & 0 & m_2 (l_1 \cos \theta_2 + l_2) & -m_2 [g \sin(\theta_1 + \theta_2) + l_1 \dot{\theta}_1^2 \sin \theta_2] \\
0 & -1 & -m_2 l_1 \sin \theta_2 & -m_2 [g \cos(\theta_1 + \theta_2) + l_1 \dot{\theta}_1^2 \cos \theta_2 + l_2 (\dot{\theta}_1 + \dot{\theta}_2)^2]
\end{vmatrix}
\]

\[
\ddot{\theta}_2 = \left(-g l_2 m_1 m_2 \sin \theta_1 - gl_1 m_1 m_2 \cos \theta_2 \sin \theta_1 + \dot{\theta}_1^2 l_1^2 m_1 m_2 \sin \theta_2 + \dot{\theta}_1 l_1^2 m_2^2 \sin \theta_2 + 2 \dot{\theta}_1 \dot{\theta}_2 l_2 m_2 \sin \theta_2 + \dot{\theta}_2^2 l_2^2 m_2 \sin \theta_2 + 2 \dot{\theta}_1 \dot{\theta}_2 l_2 m_2 \cos \theta_2 \sin \theta_2 + \dot{\theta}_2^2 l_2 m_2 \cos \theta_2 \sin \theta_2 + \dot{\theta}_2^2 l_2^2 m_2 \cos \theta_2 \sin \theta_2 + gl_2 m_2^2 \cos(\theta_1 + \theta_2) \sin \theta_2 +
\right.

\[
gl_1 m_2^2 \cos \theta_2 \cos(\theta_1 + \theta_2) \sin \theta_2 + \dot{\theta}_1 \dot{\theta}_2 l_2 m_2^2 \sin \theta_2 + gl_2 m_2^2 \sin(\theta_1 + \theta_2) + gl_1 m_1 m_2 \sin(\theta_1 + \theta_2) +
\right. 

\left. gl_1 m_2^2 \sin^2 \theta_2 \sin(\theta_1 + \theta_2) / \left(-l_1 l_2 m_1 m_2 - l_1 l_2 m_2^2 \sin^2 \theta_2\right) \right)
\]
2. Example – System of Particles (2 particles in 2D)

Behavior of the angle $\theta_1$ as a function of the time and trajectory accomplished by the mass $B$.

Behavior of angle $\theta_2$ as a function of the time and trajectory accomplished by the mass $A$.

$$\ddot{x}(t) = f(\dot{x}(t), x(t))$$
3. Example – Rigid Body in 3D (six equations)

Free-Body Diagram

Three Angular Motions – EULER’s ANGLES (rotations: Z-X-Z)

Non-Slip (Stick) Condition on Point O
3. Example – Rigid Body in 3D (six equations)

Free-Body Diagram

Three Angular Motions – EULER’s ANGLES (rotations: Z-X-Z)

Newton:

$$\sum_{B_2} F = \underline{m}_{B_2} \underline{v}^* + m_{B_2} \underline{a}^*$$

Euler:

$$\sum_{B_2} M_O = B_2 I_O \frac{d}{dt} (B_2 \omega) + B_2 \Omega \times \left( B_2 I_O \cdot B_2 \omega \right) + B_2 \underline{r}_{O-CM} \times m_{B_2} \underline{a}_O$$
3. Example – Rigid Body in 3D (six equations)
(three equations of motion + three reaction forces)

\[
\ddot{\mathbf{x}}(t) = f(\dot{\mathbf{x}}(t), \mathbf{x}(t))
\]

- Precession

\[
\ddot{\varphi} = \frac{\dot{\theta}}{\sin \theta \cdot (C_M I_{xx} + mh^2)} \cdot [\dot{\varphi} \cdot C_M I_{zz} + \varphi \cos \theta \cdot (C_M I_{zz} - C_M I_{xx} - C_M I_{yy} - 2mh^2)]
\]

- Nutation

\[
\ddot{\theta} = \frac{\sin \theta}{C_M I_{xx} + mh^2} \cdot \{mgh + \dot{\varphi} \cdot [-C_M I_{zz} \dot{\varphi} + \varphi \cos \theta \cdot (C_M I_{yy} - C_M I_{zz} + mh^2)]\}
\]

- Spin

\[
\ddot{\phi} = \frac{\dot{\theta}}{\tan \theta \cdot (C_M I_{yy} + mh^2)} \cdot [-C_M I_{zz} \dot{\phi} + \varphi \cos \theta \cdot (C_M I_{xx} + C_M I_{yy} - C_M I_{zz} + 2mh^2)]
\]
\[
+ \frac{1}{C_M I_{zz}} \cdot [M + \dot{\varphi} \dot{\theta} \sin \theta \cdot (C_M I_{xx} - C_M I_{yy} + C_M I_{zz})]
\]
2. Example – Rigid Body in 3D (six equations)
(three equations of motion + three reaction forces)

- Component $R_x$ of the dynamic reaction force:

$$R_x = \cos \varphi \left[ m \dot{\theta} h(\dot{\varphi} \cos \theta + \dot{\phi}) + mh(\ddot{\varphi} \sin \theta + \dot{\varphi} \dot{\theta} \cos \theta) \right] +$$

$$+ \sin \varphi \left[ -mg \sin \theta - m\dot{\varphi} h \sin \theta(\dot{\varphi} \cos \theta + \dot{\phi}) + mh\ddot{\theta} \right]$$

- Component $R_y$ of the dynamic reaction force:

$$R_y = \cos \theta \sin \varphi \left[ m \dot{\theta} h(\dot{\varphi} \cos \theta + \dot{\phi}) + mh(\ddot{\varphi} \sin \theta + \dot{\varphi} \dot{\theta} \cos \theta) \right] +$$

$$+ \cos \theta \cos \varphi \left[ mg \sin \theta + m\dot{\varphi} h \sin \theta(\dot{\varphi} \cos \theta + \dot{\phi}) - mh\ddot{\theta} \right] +$$

$$+ \sin \theta \left[ -mg \cos \theta + mh(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) \right]$$

- Component $R_z$ of the dynamic reaction force:

$$R_z = \sin \theta \sin \varphi \left[ m \dot{\theta} h(\dot{\varphi} \cos \theta + \dot{\phi}) + mh(\ddot{\varphi} \sin \theta + \dot{\varphi} \dot{\theta} \cos \theta) \right] +$$

$$+ \sin \theta \cos \varphi \left[ mg \sin \theta + m\dot{\varphi} h \sin \theta(\dot{\varphi} \cos \theta + \dot{\phi}) - mh\ddot{\theta} \right] -$$

$$- \cos \theta \left[ -mg \cos \theta + mh(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) \right]$$

The components $R_x$ and $R_y$, which can be calculated as a function of the top movements (precession, nutation and spin) are time-depending, i.e. $R_x(t) \neq R_y(t)$. In order to avoid slipping, the friction force has to be bigger than the resultant force $\sqrt{R_x^2 + R_y^2}$. Mathematically, $\sqrt{R_x^2 + R_y^2} \leq \mu ||R_z||$. 

- weight force $\mathbf{P}$, represented with help of the inertial reference frame

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & -mg \end{bmatrix}^T$$

- Reaction force $\mathbf{R}$, which is represented with help of the inertial reference frame

$$\mathbf{R} = \begin{bmatrix} R_x & R_y & R_z \end{bmatrix}^T$$
2. Example – Rigid Body in 3D (six equations)

CASE (a)

INITIAL CONDITIONS:

\( \psi : = 0.0; \)
\( \theta : = (15/180)\pi \times 2 \times 3.14159; \)
\( \phi : = 0.0; \)
\( \dot{\psi} : = (200/180)\pi \times 2 \times 3.14159; \)
\( \dot{\theta} : = 0.0; \)
\( \dot{\phi} : = 7 \times 2 \times 3.14159; \)
\( M : = -0.025; \)

Trajectory of the top center of mass projected into the XY-plane as a function of the time, until the disk touches the floor – Initial conditions given in the case (a).

Behavior of the height of the top center of mass as a function of the time, until the disk touches the floor – Initial conditions given in the case (a).
3. Example – Rigid Body in 3D (six equations)

**CASO (c)**

**INITIAL CONDITIONS:**

\[
\begin{align*}
&\text{psi} := 0.0; \\
&\text{teta} := (5/180) \times 2 \times 3.14159; \\
&\text{phi} := 0.0; \\
&\text{Dpsi} := 0.0; \\
&\text{Dteta} := 0.0; \\
&\text{Dphi} := 40 \times 2 \times 3.14159; \\
&\text{Mz} := -0.001;
\end{align*}
\]

**CASO (d)**

**INITIAL CONDITIONS:**

\[
\begin{align*}
&\text{psi} := 0.0; \\
&\text{teta} := (5/180) \times 2 \times 3.14159; \\
&\text{phi} := 0.0; \\
&\text{Dpsi} := 0.0; \\
&\text{Dteta} := 0.0; \\
&\text{Dphi} := 7 \times 2 \times 3.14159; \\
&\text{Mz} := -0.001;
\end{align*}
\]
4. Example – Linearization and Application to Rotating Machines

\[ M\ddot{x}(t) + (D + G(\Omega))\dot{x}(t) + Kx(t) = f(t) \]

\[ \ddot{x}(t) = f(\dot{x}(t), x(t)) \]